

# ECE 741 / 841

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## Contradiction

A formula  $p \wedge \neg p$  is said to be a contradiction.

This is because a statement can not be valid and invalid at the same time.

In general, formulas of the form  $\phi \wedge \neg \phi$  are represented by the symbol  $\perp$  and it is called bottom.

The dual of  $\perp$  is  $\top$  and it is called top.

## Basic Rules for Natural Deduction, cont.

If we have an contradiction in our premises, we can infer  $\perp$ .

$$\frac{\phi}{\perp} \quad \frac{\neg\phi}{\perp}$$

From  $\perp$ , any formula  $\phi$  can be inferred.

$$\frac{\perp}{\phi} \quad \perp e$$

## Example

Construct a proof of,

$$\neg p \vee q \vdash p \rightarrow q$$

## Example

Start with  $\neg p \vee q, p \vdash q$

- |    |                   |                |    |     |                               |
|----|-------------------|----------------|----|-----|-------------------------------|
| 1a | $\neg p$          | premise        | 1b | $q$ | premise                       |
| 2a | $p$               | premise        | 2b | $p$ | premise                       |
| 3a | $\perp$           | $\neg e$ 1a,2a | 3b | $q$ | from 1b                       |
| 4a | $q$               | $\perp e$ 3a   |    |     |                               |
| 5  | $q$               |                |    |     | by $\vee e$ on 4a,3b          |
| 6  | $p \rightarrow q$ |                |    |     | by $\rightarrow i$ on 5,2a,2b |

## Rules for Natural Deduction, cont.

Proof by contradiction:

If we assume that a formula is true and that assumption lead us to  $\perp$ , then we can conclude that the assumption is false.

$$\frac{\phi \vdash \perp}{\neg\phi} \text{ i}$$

If we assume that a formula is false and that assumption lead us to  $\perp$ , then we can conclude that the assumption is true.

$$\frac{\neg\phi \vdash \perp}{\phi} \text{ RAA (reductio ad absurdum)}$$

## Two Examples

Construct a proof of,

$$\neg p \vee \neg q \vdash \neg(p \wedge q)$$

and

$$\neg(p \wedge q) \vdash \neg p \vee \neg q$$

## First Proof

Proof by contradiction. Assume not  $\neg(p \wedge q)$

1a	$\neg p$	premise	1b	$\neg q$	premise
2a	$\neg\neg(p \wedge q)$	assump.	2b	$\neg\neg(p \wedge q)$	assump.
3a	$p \wedge q$	$\neg\neg e$ 2a	3b	$p \wedge q$	$\neg\neg e$ 2b
4a	$p$	$\wedge e$ 3a	4b	$q$	$\wedge e$ 3b
5a	$\perp$	$\neg e$ 1a, 4a	5b	$\perp$	$\neg e$ 1b, 5b
6	$\perp$	$\vee e$ 6a, 6b			
7	$\neg(p \wedge q)$	RAA 2, 6			
	$\neg p \vee \neg q$	$\vdash \neg(p \wedge q)$			

## Second Proof

Again, proof by contradiction. Assume not  $\neg p \vee \neg q$

- 1       $\neg(p \wedge q)$       premise
- 2       $\neg(\neg p \vee \neg q)$       assumption

Must prove these lemmas before continuing:

lemma dM1a  $\neg(\neg p \vee \neg q) \vdash p$

lemma dM1b  $\neg(\neg p \vee \neg q) \vdash q$

- 1       $\neg(\neg p \vee \neg q)$       premise
- 2       $\neg p$       assumption
- 3       $\neg p \vee \neg q$        $\vee i \ 2$
- 4       $\perp$       (cont.) 1,3 ( $\neg e$ )
- 5       $p$       RAA 2,4

## Second Proof, cont.

$\neg(p \wedge q) \vdash \neg p \vee \neg q$

- 1       $\neg(p \wedge q)$       premise
- 2       $\neg(\neg p \vee \neg q)$       assumption
- 3       $p$                           by lemma dM1a
- 4       $q$                           by lemma dM1b
- 5       $p \wedge q$                  $\wedge i \ 3,4$
- 6       $\perp$                       cont. 1,5 ( $\neg e$ )
- 7       $(\neg p \vee \neg q)$       RAA 2,6

## Caution

Construct a proof of,

$\phi \vdash \psi$			
1	$\phi$	premise	
2	$\neg\psi$	assumption	
:	:		
n	$\perp$	cont. ( $\neg e$ )	
n+1	$\psi$	$\perp e 2, n$	
n+2	$\chi$	$\perp e n$	
			$\phi \vdash \psi$
			$\phi, \neg\psi \vdash \chi$
			$\phi \not\vdash \chi$

## Provable Equivalence

$\phi$  and  $\psi$  are provably equivalent iff (if and only if)

$\phi \vdash \psi$  and

$\psi \vdash \phi$

That is, if we can prove  $\psi$  using  $\phi$  as premise and  $\psi$  using  $\phi$  as premise. This is denoted,

$\phi \dashv\vdash \psi$

since we have shown that

$\neg p \vee \neg q \vdash \neg(p \wedge q)$  and  $\neg(p \wedge q) \vdash \neg p \vee \neg q$

These two formulas are provably equivalent.

deMorgan-1  $\neg p \vee \neg q \dashv\vdash \neg(p \wedge q)$

## Law of Excluded Middle (Proof by Cases)

$$\frac{}{\phi \vee \neg\phi} \text{LEM}$$

Example: Construct a proof of  $p \rightarrow q \vdash \neg p \vee q$

## Proof

- 1  $p \rightarrow q$  premise
- 2  $p \vee \neg p$  LEM
- 3a  $p$
- 3b  $\neg p$
- 4a  $q$  MP 1,3a
- 4b  $\neg p \vee q$   $\vee i$  3b
- 5a  $\neg p \vee q$   $\vee i$  4a
- 6  $\neg p \vee q$   $\vee e$  5a,4b

## Semantics of Propositional Logic

So far, we have been dealing with syntactic manipulation to infer the validity of a formula from a set of premises.

Another relation between premisses and conclusion is determined by semantics and expressed by,

$$\phi \models \psi$$

It is determined by the meaning of  $\phi$  and  $\psi$ . In this case, the meaning of these formulas is *true* or *false*. True or false is represented by T and F, respectively.

If  $\psi$  evaluates to T whenever  $\phi$  evaluates to T, then we denote this relation by  $\phi \models \psi$  and call it semantics entailment.

## Evaluating WFF

In order to evaluate formulas (wff), we must give meaning to the syntactic connectives  $\neg$ ,  $\vee$ ,  $\wedge$ , and  $\rightarrow$ .

These connective have the usual meaning. For example, formula  $p \wedge q$  evaluates to  $T$  iff  $p$  evaluates to  $T$  and  $q$  evaluates to  $T$ .

## Meaning of the $\rightarrow$ Connective

$\phi$	$\psi$	$\phi \rightarrow \psi$
F	F	T
F	T	T
T	F	F
T	T	T

## Soundness of Propositional Logic

Using natural deduction rules, we can infer the validity of a formula  $\psi$  from a premise  $\phi$ .

If using rules of deduction we conclude  $\phi \vdash \psi$ , is it possible that  $\psi$  will evaluate to  $\text{F}$  when  $\phi$  evaluates to  $\text{T}$ ? If the logic is sound, then that is not the case.

We say that the logic is sound if whenever  $\phi \vdash \psi$  then  $\phi \models \psi$ .

## Practical Aspect of Soundness

When a logic is sound, we can establish the validity of a formula by evaluating the formula over its domain.

$$\text{deMorgan-1 } \neg p \vee \neg q \dashv \vdash \neg(p \wedge q)$$

$p$	$\neg p$	$q$	$\neg q$	$\neg p \vee \neg q$	$\neg(p \wedge q)$
F	T	F	T	T	T
F	T	T	F	T	T
T	F	F	T	T	T
T	F	F	T	F	F

## Semantics Equivalence, Satisfiability and Validity

Two formulas  $\phi$  and  $\psi$  are semantically equivalent if they have the same meaning. That is, if they evaluate to the same value. Semantic equivalence is represented thus,

$$\phi \equiv \psi$$

A formula is satisfiable if there is at least one assignment of truth values to its propositional atoms such that it evaluates to T.

A formula is valid (a tautology) if for all assignments of truth values to its propositional atoms the formula evaluates to true.

# **Homework**